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Labeling the r-path with a condition at distance two

John P. Georges, David W. Mauro *, Yan Wang [1]

Department of Mathematics, Trinity College, Hartford, CT 06013, USA

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ABSTRACT

For integer $r \geq 2$, the infinite r-path $P_\infty(r)$ is the graph on vertices $\ldots v_{-3}, v_{-2}, v_{-1}, v_0, v_1, v_2, v_3 \ldots$ such that v_s is adjacent to v_t if and only if $|s-t| \leq r-1$. The r-path on n vertices is the subgraph of $P_\infty(r)$ induced by vertices $v_0, v_1, v_2, \ldots, v_{n-1}$. For non-negative reals x_1 and x_2, a λ_{x_1,x_2}-labeling of a simple graph G is an assignment of non-negative reals to the vertices of G such that adjacent vertices receive reals that differ by at least x_1, vertices at distance two receive reals that differ by at least x_2, and the absolute difference between the largest and smallest assigned reals is minimized. With $\lambda_{x_1,x_2}(G)$ denoting that minimum difference, we derive $\lambda_{x_1,x_2}\left(P_n(r)\right)$ for $r \geq 3$, $1 \leq n \leq \infty$, and $\frac{x_1}{x_2} \in [2, \infty]$. For $\frac{x_1}{x_2} \in [1, 2]$, we obtain upper bounds on $\lambda_{x_1,x_2}\left(P_\infty(r)\right)$ and use them to give $\lambda_{x_1,x_2}\left(P_\infty(r)\right)$ for $r \geq 5$ and $\frac{x_1}{x_2} \in \left[1, \frac{2r-2}{2r-3}\right] \bigcup \left[\frac{4}{3}, 2\right]$. We also determine $\lambda_{x_1,x_2}\left(P_\infty(3)\right)$ and $\lambda_{x_1,x_2}\left(P_\infty(4)\right)$ for all $\frac{x_1}{x_2} \in [1, 2]$.

1. Introduction

The well-known distance-constrained graph labeling problem was introduced by Griggs and Yeh [10] as the graph-theoretic analog of the channel assignment problem [11] in which one seeks the shortest possible interval from which to allot frequencies to transmitters subject to constraints determined by distances among those transmitters. Letting the vertices of the simple graph $G = (V, E)$ denote transmitters and letting an assignment of non-negative reals to the vertices of G represent the assignment of frequencies, researchers at first sought the elimination of transmission interference by requiring that the distance between two vertices be inversely related to the absolute difference between their assigned numbers. Lately, however, this inverse relationship has been relaxed as authors (notably Griggs and Jin in [7]) have considered distance-constrained vertex labelings in a more general context. We thus begin with a general definition of a distance-constrained vertex labeling, noting that the condition $x_1 \geq x_2$ addresses the goal of the elimination of transmission interference.

Definition 1.1. For simple graph $G = (V, E)$ and non-negative reals x_1, x_2, an $L(x_1, x_2)$-labeling of G is a function $L : V \to \mathbb{R}^+$ (the set of non-negative reals) such that for all vertices v, w in V,

(i) $|L(v) - L(w)| \geq x_1$ if v and w are adjacent, and
(ii) $|L(v) - L(w)| \geq x_2$ if v and w are at distance two.

The λ_{x_1,x_2}-number of G, denoted by $\lambda_{x_1,x_2}(G)$, is the minimum span among the $L(x_1, x_2)$-labelings of G, and any $L(x_1, x_2)$-labeling of G that achieves the minimum span is called a λ_{x_1,x_2}-labeling of G. $\square$

* Corresponding author. Tel.: +1 860 297 2292; fax: +1 860 987 6239.
E-mail address: david.mauro@trincoll.edu (D.W. Mauro).

1 Presently at Department of Mathematics, Millsaps College, Jackson, MS 39210, USA.

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Numerous authors have investigated $\lambda_{j,k}(G)$ for integers $1 \le k \le j$ and various G including paths, cycles, trees, unit interval graphs, products of complete graphs, lattices, hypercubes, and generalized Petersen graphs. Others have considered the relationship between $\lambda_{j,k}(G)$ and graph invariants of G such as maximum degree, size, path covering number, chromatic number, and packing number (for surveys, see [2,9,16]). Establishing what is perhaps the best-known open question of the field, Griggs and Yeh [10] have conjectured that for any graph G, $\lambda_{2,1}(G) \le \Delta^2(G)$. Recent results include the proof of the conjecture for G with sufficiently large Δ [12] and an improved upper bound of $\lambda_{p,1}(G)$ for integer $p \ge 2$ and G with arbitrary maximum degree Δ [5].

For any non-negative reals x_1 and x_2, it is clear that if c is a positive constant and L is an $L(x_1, x_2)$-labeling of G with span $sp(L)$, then cL is an $L(cx_1, cx_2)$-labeling of G with span $c\big(sp(L)\big)$. It is thus easy to verify that $\lambda_{cx_1,cx_2}(G) = c\lambda_{x_1,x_2}(G)$, particularly implying $\lambda_{x_1,x_2}(G) = x_2\lambda_{\frac{x_1}{x_2},1}(G)$. Therefore, to study the λ_{x_1,x_2}-numbers of graph G, it suffices to study the $\lambda_{x,1}$-numbers of G for $x \ge 0$. Moreover, it is shown in [7] that $\lambda_{x,1}(G)$ is a continuous function of x on $\mathbb{R}^+$. Since every non-negative real is the limit of a sequence of non-negative rationals, it follows that for any closed interval $[a, b]$ of non-negative reals, $\lambda_{x,1}(G)$ on $[a, b]$ is completely determined by $\lambda_{q,1}(G)$ on $\mathbb{Q} \bigcap]a, b[$, where $]a, b[$ denotes the open interval from a to b. For this reason, it suffices to consider $\lambda_{j,k}(G)$ for non-negative integers j, k in the investigation of $\lambda_{x,1}(G)$.

In this paper, for integers $0 \le k \le j$, we investigate the $\lambda_{j,k}$-number of $P_n(r)$ and $P_\infty(r)$ where $P_n(r)$ and $P_\infty(r)$ are the graphs with respective vertex sets $\{v_0, v_1, v_2, \ldots, v_n\}$ and $\{\ldots v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\}$ such that v_i, v_m are adjacent if and only if $1 \le |i - m| \le r - 1$. We note that the conventional finite and infinite paths are respectively $P_n(2)$ and $P_\infty(2)$, and that $P_n(r)$ is isomorphic to K_n for $1 \le n \le r$. We also point out that although not all physical deployments of transmitters can be accurately modelled by a graph, [2] the graphs $P_n(r)$ and $P_\infty(r)$ have the particular virtue that they accurately model a physical deployment of transmitters along a straight line. Other graphs that model realistic deployments include various lattices, the $\lambda_{x,1}$-numbers of which have recently been studied by Calamoneri [1], Griggs and Jin [6] and Král' and Skoda [14].

In Section 2, we give several preliminary results and definitions that will facilitate the discussions and derivations throughout the paper. In Section 3, we derive $\lambda_{j,k}\big(P_n(r)\big)$ for all $n \le \infty$ and all $\frac{j}{k} \ge 2$. The case $1 \le \frac{j}{k} < 2$ appears to be more difficult. In Section 4, we use some of the properties of $\lambda_{x,1}(G)$ (recently proved in [7]) to derive $\lambda_{j,k}\big(P_\infty(r)\big)$ for $\frac{j}{k} \in [1, \frac{2r-2}{2r-3}] \bigcup [\frac{4}{3}, 2]$. We also derive $\lambda_{j,k}\big(P_\infty(3)\big)$ and $\lambda_{j,k}\big(P_\infty(4)\big)$ for all $\frac{j}{k} \in [1, 2]$.

2. Definitions and preliminary results

We begin with the following theorem of Griggs and Jin (later extended by Král' [13] to include distance constraints beyond 2).

Theorem 2.1 (*[7]*). *Let $G = (V, E)$ be a simple finite or infinite graph of bounded degree. Then for every real $x \ge 0$, there exists a $\lambda_{x,1}$-labeling L of G such that for every vertex $v \in V(G)$, $L(v) = m_v x + b_v$ for some non-negative integers m_v, b_v. It thus follows that $\lambda_{x,1}(G) = mx + b$ for some non-negative integers m, b. Moreover, as a function of x, $\lambda_{x,1}(G)$ is non-decreasing, continuous, and piecewise linear on $\mathbb{R}^+$ such that for every maximal interval I on which $\lambda_{x,1}(G)$ is linear, $\lambda_{x,1}(G) = m_I x + b_I$ for some non-negative integers m_I, b_I.* $\square$

By this theorem, for every graph G and all non-negative reals x_1, x_2, there exists a λ_{x_1,x_2}-labeling L of G such that every label of L is of the form $mx_1 + bx_2$ for some non-negative integers m and b. Additionally, it is clear that if L is an $L(x_1, x_2)$-labeling of G, then for any fixed integer c, $L - c$ is an $L(x_1, x_2)$-labeling of G such that $sp(L) = sp(L - c)$. Therefore, without loss of generality and unless stated otherwise, we shall confine our attention to only those λ_{x_1,x_2}-labelings under which the minimum assigned label is 0 and each assigned label is of the form $mx_1 + bx_2$ for some non-negative integers m and b. Such λ_{x_1,x_2}-labelings shall be called *normalized*.

Also by Theorem 2.1, it is clear that if x is irrational and $\lambda_{x,1}(G) = mx + b$ for non-negative integers m, b, then m and b are unique. Hence by the continuity of $\lambda_{x,1}(G)$, the points of non-differentiability of $\lambda_{x,1}(G)$ cannot be irrational. Moreover, owing to the density of the rationals in the reals, $\lambda_{x,1}(G) = mx + b$ for all x in the interval $[z_1, z_2]$ if and only if $\lambda_{j,k}(G) = mj + bk$ for all positive integers j, k such that $\frac{j}{k} \in]z_1, z_2[$. As indicated in the previous section, we therefore concentrate our discussion on $\lambda_{j,k}(G)$ for positive integers j, k.

The following three theorems review the existing results on $P_n(2)$.

Theorem 2.2 (*[3]*). *For positive integers j and k such that $\frac{j}{k} \ge 2$,*

$$\lambda_{j,k}\big(P_n(2)\big) = \begin{cases} 0 & \text{if } n = 1 \\ j & \text{if } n = 2 \\ j + k & \text{if } n = 3, 4 \\ j + 2k & \text{if } 5 \le n \le \infty. \end{cases}$$

[2] For example, consider a network of only two transmitters such that their assigned frequencies must satisfy a distance two condition. There is clearly no corresponding graph with order two such that the two vertices are distance two apart.

Theorem 2.3 ([3]). *For positive integers j and k such that* $1 \le \frac{j}{k} \le 2$,

$$\lambda_{j,k}(P_n(2)) = \begin{cases} 0 & \text{if } n = 1 \\ j & \text{if } n = 2 \\ j+k & \text{if } n = 3, 4 \\ 2j & \text{if } 5 \le n \le \infty. \end{cases}$$

Theorem 2.4 ([8]). *For non-negative integers j and k such that* $0 \le \frac{j}{k} \le 1$,

$$\lambda_{j,k}(P_n(2)) = \begin{cases} 0 & \text{if } n = 1 \\ j & \text{if } n = 2 \\ k & \text{if } 0 \le \dfrac{j}{k} \le \dfrac{1}{2}, n = 3 \\ 2j & \text{if } \dfrac{1}{2} \le \dfrac{j}{k} \le 1, n = 3 \\ j+k & \text{if } n = 4, 5, 6 \\ j+k & \text{if } 0 \le \dfrac{j}{k} \le \dfrac{1}{2}, n \ge 7 \\ 3j & \text{if } \dfrac{1}{2} \le \dfrac{j}{k} \le \dfrac{2}{3}, n \ge 7 \\ 2k & \text{if } \dfrac{2}{3} \le \dfrac{j}{k} \le 1, n \ge 7. \end{cases}$$

Since $P_n(r)$ is isomorphic to K_n for $n \le r$, the following is immediate.

Theorem 2.5. *For non-negative integers j and k such that* $0 \le \frac{j}{k}$ *and for integers n and r such that* $1 \le n \le r$, $\lambda_{j,k}(P_n(r)) = j(n-1)$. □

It is clear that Theorems 2.2–2.5 permit us to restrict our attention to the cases $n \ge r+1$ and $r \ge 3$.
We close this section with the derivation of $\lambda_{j,k}(P_n(r))$ for $j = k = 1$.

Theorem 2.6. *For a positive integer n or* $n = \infty$,

$$\lambda_{1,1}(P_n(r)) = \begin{cases} n-1 & \text{if } 1 \le n \le 2r-1 \\ 2r-2 & \text{if } 2r \le n \le \infty \end{cases}$$

Proof. For $r = 2$ or $n \le r$, the result follows from Theorems 2.3 and 2.5. So we assume $r \ge 3$ and $n > r$ (although the argument below applies if $r = 2$).

Let L denote a $\lambda_{1,1}$-labeling of $P_n(r)$. If $r < n \le 2r-1$, then $P_n(r)$ has diameter two, and so $\lambda_{1,1}(P_n(r)) = n-1$.

Now suppose $2r \le n \le \infty$. Since $P_{2r-1}(r)$ is a subgraph of $P_n(r)$, it follows that $\lambda_{1,1}(P_{2r-1}(r)) \le \lambda_{1,1}(P_n(r))$. Thus $2r-2 \le \lambda_{1,1}(P_n(r))$. But it is easy to see that the labeling given by $L^*(v_i) = i \bmod (2r-1)$ is an $L(1,1)$-labeling of $P_n(r)$ with span $2r-2$. So $\lambda_{1,1}(P_n(r)) \le 2r-2$, giving the result. □

3. On $\lambda_{j,k}(P_n(r))$ for $\frac{j}{k} \ge 2$

In this section, we prove the following.

Theorem 3.1. *For* $\frac{j}{k} \ge 2$,

$$\lambda_{j,k}(P_n(r)) = \begin{cases} (r-1)j+k & \text{if } r+1 \le n \le 2r \\ (r-1)j+2k & \text{if } 2r+1 \le n \le 3r \\ (r-1)j+\gamma k & \text{if } \exists \gamma \in \mathbb{N} \text{ such that} \\ & \quad \gamma r+1 \le n \le (\gamma+1)r, \dfrac{j}{k} \ge \gamma, 3 \le \gamma \le r \\ rj & \text{if } \exists \gamma \in \mathbb{N} \text{ such that} \\ & \quad \gamma r+1 \le n \le (\gamma+1)r, \dfrac{j}{k} < \gamma, 3 \le \gamma \le r \\ (r-1)j+rk & \text{if } (r+1)r < n \le \infty, \dfrac{j}{k} > r \\ rj & \text{if } (r+1)r < n \le \infty, \dfrac{j}{k} \le r. \end{cases}$$

We begin by observing that since $\frac{j}{k} \geq 2$, an $L(j,k)$-labeling of $P_\infty(r)$ can be formed by assigning the labels

$$0, j, 2j, \ldots, rj, k, k+j, k+2j, \ldots, k+(r-1)j$$

to the respective vertices

$$v_0, v_1, v_2, \ldots, v_{2r+1},$$

and then repeating that pattern of assignments. Since the span of this labeling is rj and since $P_n(r)$ is a subgraph of $P_\infty(r)$ for $1 \leq n \leq \infty$, we thus have

Lemma 3.2. *For $1 \leq n \leq \infty$, $\lambda_{j,k}\big(P_n(r)\big) \leq rj$.* □

We note that for $j = 2$ and $k = 1$, the above labeling is identical to the labeling scheme for unit interval graphs provided by Sakai [15].

We now turn our attention to finite n. In this case, for $0 \leq i \leq n-1$, it will be useful to express vertex v_i as $v_{\alpha,\beta}$ where $i = \alpha r + \beta$, $0 \leq \beta \leq r-1$. Under this notation, with $n = ar + b$, the r-path $P_n(r)$ can be represented in array form as indicated:

$$\begin{array}{cccccc}
v_{0,0} & v_{0,1} & v_{0,2} & \cdots & \cdots & v_{0,r-1} \\
v_{1,0} & v_{1,1} & v_{1,2} & \cdots & \cdots & v_{1,r-1} \\
v_{2,0} & v_{2,1} & v_{2,2} & \cdots & \cdots & v_{2,r-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_{a-1,0} & v_{a-1,1} & v_{a-1,2} & \cdots & \cdots & v_{a-1,r-1} \\
v_{a,0} & v_{a,1} & v_{a,2} & \cdots & v_{a,b-1}. &
\end{array}$$

For $0 \leq c \leq r-1$, the number of vertices in column c will be denoted by c^*, and for $0 \leq \alpha, \alpha' \leq c^*-1$, vertices $v_{\alpha,c}$ and $v_{\alpha',c}$ will be called *column-adjacent* if and only if $|\alpha - \alpha'| = 1$. For $0 \leq p \leq q \leq c^*-1$, the set of vertices $\{v_{p,c}, v_{p+1,c}, v_{p+2,c}, v_{p+3,c}, \ldots, v_{q,c}\}$ shall be denoted by $S_p^q(c)$. Thus $S_0^{c^*-1}(c)$ is the set of vertices in column c.

We make the following observations and definitions.

Observation 3.3. For finite $n \geq 1$, vertices $v_{\alpha,\beta}$ and $v_{\alpha',\beta'}$ in $V\big(P_n(r)\big)$ are adjacent if and only if $1 \leq |(r\alpha+\beta)-(r\alpha'+\beta')| \leq r-1$. If vertices $v_{\alpha,\beta}$ and $v_{\alpha',\beta'}$ are adjacent, then without loss of generality, $\beta - \beta' \geq 1$ and $0 \leq \alpha' - \alpha \leq 1$. □

Observation 3.4. For finite $n \geq 1$, vertices $v_{\alpha,\beta}$ and $v_{\alpha',\beta'}$ in $V\big(P_n(r)\big)$ are at distance two if and only if $r \leq |(r\alpha+\beta)-(r\alpha'+\beta')| \leq 2r-2$. If vertices $v_{\alpha,\beta}$ and $v_{\alpha',\beta'}$ are at distance two, then without loss of generality, either $\alpha - \alpha' = 1$ and $\beta' - \beta \leq 0$, or $\alpha - \alpha' = 2$ and $\beta' - \beta \geq 2$. □

Definition 3.5. Let a, r and n be positive integers such that $n = ar+1$. Let σ be a permutation of $\{0, 1, 2, \ldots, r-1\}$. Then for $0 \leq i \leq \min\{a-1, r-1\}$, the core of column $\sigma(i)$ of $P_{ar+1}(r)$, denoted by $Core_{a,r}(\sigma(i))$, is defined recursively as follows:

For $i = 0$, $Core_{a,r}(\sigma(i)) = S_0^{\sigma(i)^*-1}(\sigma(i))$.

For $i \geq 1$, let p and q be such that $Core_{a,r}(\sigma(i-1)) = S_p^q(\sigma(i-1))$. Then

$$Core_{a,r}(\sigma(i)) = \begin{cases} S_p^{q-1}(\sigma(i)) & \text{if } \sigma(i) > \sigma(i-1) \\ S_{p+1}^{q}(\sigma(i)) & \text{if } \sigma(i) < \sigma(i-1). \end{cases}$$

The subscript $_{a,r}$ may be suppressed when there is no possibility for confusion.

Observation 3.6. For $0 \leq i < \min\{a-1, r-1\}$, $|Core_{a,r}(\sigma(i+1))| = |Core_{a,r}(\sigma(i))| - 1$, implying $|Core_{a,r}(\sigma(i))| \geq 2$ for $i < \min\{a-1, r-1\}$ and $|Core_{a,r}(\sigma(\min\{a-1, r-1\}))| \geq 1$. Additionally, each vertex in $Core_{a,r}(\sigma(i))$ is adjacent to two column-adjacent vertices in $Core_{a,r}(\sigma(i-1))$. □

Definition 3.7. Let j be a fixed positive integer. Then for each non-negative integer l, the set of consecutive integers $S_{j,l} = \{lj, lj+1, lj+2, \ldots, (l+1)j-1\}$ shall be called *label$_j$-class l*. If L is an $L(j,k)$-labeling of graph G, then we will say that vertex v has label$_j$-class l under L if and only if $L(v) \in S_{j,l}$. □

Note that under any $L(j,k)$-labeling, no two adjacent vertices have labels in the same label$_j$-class.

Lemma 3.8. *For finite $n \geq 1$, let L denote an $L(j,k)$-labeling of $P_n(r)$ with span less than rj. Then the labels of the vertices in column c under L must be in the same* label$_j$*-class.*

Proof. It suffices to show that $L(v_{\alpha,c})$ and $L(v_{\alpha+1,c})$ are in the same label$_j$-class, $0 \le \alpha \le c^* - 2$.

Consider the sequence of $r+1$ vertices

$$v_{\alpha,c}, v_{\alpha,c+1}, v_{\alpha,c+2}, \ldots, v_{\alpha,r-1}, v_{\alpha+1,0}, v_{\alpha+1,1}, \ldots, v_{\alpha+1,c-1}, v_{\alpha+1,c}.$$

Due to the distance one condition, the first r vertices of the sequence have distinct label$_j$-classes under L and the last r vertices of the sequence have distinct label$_j$-classes under L. Since the span of L is less than rj, the labels assigned by L fall into exactly r distinct label$_j$-classes, and hence the result follows. □

Lemma 3.9. *Let L denote an $L(j,k)$-labeling of $P_n(r)$ with span less than rj, where $n = ar + 1$ and $1 \le a < \infty$. By Lemma 3.8, let σ be a permutation of* label$_j$*-classes* $0, 1, 2, \ldots, r-1$ *such that under L, the vertices of column $\sigma(c)$ are labeled with integers from* label$_j$*-class c for $0 \le c \le r-1$. Then*

(i) *for each i, $0 \le i \le \min\{a-1, r-1\}$, the vertices of $Core(\sigma(i))$ have labels at least $ij + ik$, and*
(ii) *for each i, $0 \le i \le \min\{a-1, r-1\}$, if $|Core(\sigma(i))| \ge 2$, then there exists a vertex in $Core(\sigma(i))$ with label at least $ij + (i+1)k$.*

Proof. (i) By induction. Clearly, the vertices of $Core(\sigma(0))$ have labels at least $0j + 0k = 0$. Thus, suppose that h is an integer less than $\min\{a-1, r-1\}$ for which the vertices of $Core(\sigma(h))$ have labels at least $hj + hk$. Since $h < \min\{a-1, r-1\}$, then $|Core(\sigma(h))| \ge 2$ by Observation 3.6 Thus, by the distance 2 condition and the inductive hypothesis, every pair of column-adjacent vertices in $Core(\sigma(h))$ includes at least one vertex with label at least $hj + (h+1)k$. But every vertex in $Core(\sigma(h+1))$ is adjacent to two column-adjacent vertices in $Core(\sigma(h))$ by Observation 3.6. Thus, every vertex in $Core(\sigma(h+1))$ is adjacent to at least one vertex with label $hj + (h+1)k$, from which (i) follows.

(ii) If $Core(\sigma(i))$ contains at least two vertices, then $Core(\sigma(i))$ contains two vertices at distance 2 with labels (by (i)) at least $ij + ik$. Thus (ii) follows. □

Lemma 3.10. *For finite n, let L denote an $L(j,k)$-labeling of $P_n(r)$ with span less than rj, and by Lemma 3.8, let σ be a permutation of the* label$_j$*-classes* $0, 1, 2, \ldots, r-1$ *such that the vertices of column $\sigma(c)$ are labeled with integers from* label$_j$*-class c under L for $0 \le c \le r-1$. Then for any vertex $v_{x,\sigma(h)}$ in column $\sigma(h)$ and integer z such that $h + z \le r-1$, there exists a vertex $v_{x',\sigma(h+z)}$ in column $\sigma(h+z)$ such that $L(v_{x',\sigma(h+z)}) \ge L(v_{x,\sigma(h)}) + zj$.*

Proof. The result follows easily from the distance one condition and an inductive argument on z. □

Lemma 3.11. *Suppose $\gamma r + 1 \le n \le (\gamma+1)r$, where $1 \le \gamma \le r$, and suppose $\lambda_k^j(P_{\gamma r+1}(r)) < rj$. Then $\lambda_{j,k}(P_n(r)) \ge (r-1)j + \gamma k$.*

Proof. Assume that $n = \gamma r + 1$.

Let L denote an arbitrary $L(j,k)$-labeling of $P_{\gamma r+1}(r)$ with span less than rj, and by Lemma 3.8 let σ be a permutation of the label$_j$-classes $0, 1, 2, \ldots, r-1$ such that the vertices of column $\sigma(i)$ are labeled with integers from label$_j$-class i, $0 \le i \le r-1$. We consider separately the three cases $1 < \gamma < r$, $1 = \gamma < r$, and $1 < \gamma = r$.

Case 1: $1 < \gamma < r$ (vacuous if $r = 2$).

Case i: $\sigma(0) = 0$. Since $|Core(\sigma(0))| = \gamma + 1$, then $|Core(\sigma(\gamma-1))| = 2$. By Lemma 3.9, $Core(\sigma(\gamma-1))$ contains a vertex with label at least $(\gamma-1)j + \gamma k$. The result follows from Lemma 3.10 with $z = r - \gamma$.

Case ii: $\sigma(0) \ne 0$. In this case, $|Core(\sigma(\gamma-1))| = 1$, or equivalently, $Core(\sigma(\gamma-1)) = \{v_{x,\sigma(\gamma-1)}\}$ for some x, $0 \le x \le \gamma - 1$. By Lemma 3.9, $L(v_{x,\sigma(\gamma-1)}) \ge (\gamma-1)j + (\gamma-1)k$. If $1 \le x \le \gamma-2$, $v_{x,\sigma(\gamma-1)}$ is adjacent to two column-adjacent vertices in column $\sigma(\gamma)$, implying the existence of a vertex in column $\sigma(\gamma)$ with label at least $\gamma j + \gamma k$. The result now follows by Lemma 3.10 with $z = r - \gamma - 1$. We thus consider the cases $x = 0$ and $x = \gamma - 1$.

Subcase a: $x = 0$. Then Definition 3.5 and the assumption of Case ii require that $0 < \sigma(0) < \sigma(1) < \sigma(2) < \cdots < \sigma(\gamma-1)$, implying $\sigma(m) = 0$ for some m, $\gamma \le m \le r-1$. If $m = \gamma$, then $v_{0,\sigma(\gamma-1)}$ is adjacent to two column-adjacent vertices in column $\sigma(\gamma) = 0$; namely, $v_{0,0}$ and $v_{1,0}$. By the distance conditions, at least one of those vertices has label at least $\gamma j + \gamma k$. The result follows by Lemma 3.10 with $z = r - \gamma - 1$. If $m > \gamma$, then by Lemma 3.10 with $z = m - \gamma$, there exists some vertex in column $\sigma(m-1)$ with label at least $(m-1)j + (\gamma-1)k$. This vertex is adjacent to two column-adjacent vertices in column $\sigma(m) = 0$, which implies that at least one of those vertices has label at least $mj + \gamma k$. The result again follows by Lemma 3.10 with $z = r - m - 1$.

Subcase b: $x = \gamma - 1$. Then by Definition 3.5, $\sigma(0) > \sigma(1) > \sigma(2) > \cdots > \sigma(\gamma-1) \ge 0$. If $\sigma(\gamma-1) = 0$, then $v_{x,\sigma(\gamma-1)} = v_{\gamma-1,0}$ and (since $\sigma(0) \ne 0$) $\gamma \ne 1$. Thus $\gamma \ge 2$, implying that $v_{\gamma-1,0}$ is adjacent to the two column-adjacent vertices $v_{\gamma-1,\sigma(\gamma)}$ and $v_{\gamma-2,\sigma(\gamma)}$. The distance conditions then imply that at least one of those two vertices has label at least $\gamma j + \gamma k$, from which the result follows by Lemma 3.10 with $z = r - \gamma - 1$. If $\sigma(\gamma-1) > 0$, then $\sigma(m) = 0$ for some m, $\gamma \le m \le r-1$. If $m = \gamma$, then $v_{\gamma-1,\sigma(\gamma-1)}$ is adjacent to two column-adjacent vertices in column $\sigma(\gamma) = 0$; namely, $v_{\gamma-1,0}$ and $v_{\gamma,0}$. By the distance conditions, at least one of those vertices has label at least $\gamma j + \gamma k$. The result follows by Lemma 3.10 with $z = r - \gamma - 1$. Thus, suppose $m > \gamma$. We have already observed that $L(v_{x,\sigma(\gamma-1)}) \ge (\gamma-1)j + (\gamma-1)k$. Hence by Lemma 3.10 with $z = m - \gamma$, there exists a vertex in $Core(\sigma(m-1))$ with label at least $(m-1)j + (\gamma-1)k$. This vertex is adjacent to two column-adjacent vertices in column $\sigma(m) = 0$, which implies that at least one of those vertices has label at least $mj + \gamma k$. The result again follows by Lemma 3.10 with $z = r - m - 1$.

Case 2: $1 = \gamma < r$. If $\sigma(0) = 0$, then either $L(v_{0,0})$ or $L(v_{1,0})$ has label in $[k, j-1)$. Thus there is a subgraph of $P_{r+1}(r)$ isomorphic to K_r with smallest label at least k, giving the result. If $\sigma(0) > 0$, then for some $m > 0$, $\sigma(m) = 0$. So $L(v_{0,\sigma(m-1)}) \geq (m-1)j$ by the definition of σ, and $v_{0,\sigma(m-1)}$ is adjacent to column-adjacent vertices $v_{0,0}$ and $v_{1,0}$, implying that at least one of those vertices has label at least $mj + k$. The result now follows by Lemma 3.10 with $z = r - m - 1$.

Case 3: $1 < \gamma = r$.

Case i: $\sigma(0) = 0$. Since $|Core(\sigma(0))| = r + 1$, then $|Core(\sigma(r-1))| = 2$. By Lemma 3.9, $Core(\sigma(r-1))$ contains a vertex with label at least $(r-1)j + rk$.

Case ii: $\sigma(0) \neq 0$. In this case, $|Core(\sigma(r-2))| = 2$. By Lemma 3.9, there exists vertex $v_{x,\sigma(r-2)}$ in $Core(\sigma(r-2))$ such that $L(v_{x,\sigma(r-2)}) \geq (r-2)j + (r-1)k$, $0 \leq x \leq r-1$. If $1 \leq x \leq r-2$, $v_{x,\sigma(r-2)}$ is adjacent to two column-adjacent vertices in column $\sigma(r-1)$, implying the existence of a vertex in column $\sigma(r-1)$ with label at least $(r-1)j + rk$. We next consider separately the cases $x = 0$ and $x = r-1$.

Subcase a: $x = 0$. Then by Definition 3.5, $0 < \sigma(0) < \sigma(1) < \sigma(2) < \cdots < \sigma(r-2)$, implying $\sigma(r-1) = 0$. Thus $v_{0,\sigma(r-2)}$ is adjacent to two column-adjacent vertices in column $\sigma(r-1) = 0$; namely, $v_{0,0}$ and $v_{1,0}$. By the distance conditions, at least one of those vertices has label at least $(r-1)j + rk$.

Subcase b: $x = r-1$. Then by Definition 3.5, $\sigma(0) > \sigma(1) > \sigma(2) > \cdots > \sigma(r-2) \geq 0$. If $\sigma(r-2) = 0$, then $v_{x,\sigma(r-2)} = v_{r-1,0}$. Thus, $v_{r-1,0}$ is adjacent to the two column-adjacent vertices $v_{r-1,\sigma(r-1)}$ and $v_{r-2,\sigma(r-1)}$, from which it follows that at least one of those two vertices has label at least $(r-1)j + rk$. If $\sigma(r-2) > 0$, then $\sigma(r-1) = 0$, implying that $v_{r-1,\sigma(r-2)}$ is adjacent to the two column-adjacent vertices $v_{r-1,0}$ and $v_{r,0}$ in column $\sigma(r-1) = 0$. By the distance conditions, at least one of those vertices has label at least $(r-1)j + rk$. □

Proof of Theorem 3.1. Suppose γ is an integer, $1 \leq \gamma \leq r$, such that $\gamma r + 1 \leq n \leq (\gamma+1)r$. We first demonstrate an $L(j,k)$-labeling L_γ of $P_n(r)$ with span $(r-1)j + \gamma k$.

Let L_γ be the integer assignment to the vertices of $P_n(r)$ such that $L_\gamma(v_{x,y}) = (r-1-y)j + xk$. Then the minimum and maximum labels, assigned to $v_{0,r-1}$ and $v_{\gamma,0}$, respectively, are 0 and $(r-1)j + \gamma k$.

To see that L_γ satisfies the distance one condition, suppose $v_{\alpha,\beta}$ and $v_{\alpha',\beta'}$ are adjacent. Then $|L_\gamma(v_{\alpha',\beta'}) - L_\gamma(v_{\alpha,\beta})| = |j(\beta - \beta') + k(\alpha' - \alpha)| \geq j$ by Observation 3.3.

To see that L_γ satisfies the distance two condition, suppose $v_{\alpha,\beta}$ and $v_{\alpha',\beta'}$ are at distance two. Then $|L_\gamma(v_{\alpha,\beta}) - L_\gamma(v_{\alpha',\beta'})| = |j(\beta' - \beta) + k(\alpha - \alpha')| \geq k$ by Observation 3.4 and the assumption $\frac{j}{k} \geq 2$.

We now proceed by considering the various values for n.

Case 1. Fix n, $r + 1 \leq n \leq 2r$. Then $\gamma r + 1 \leq n \leq (\gamma+1)r$ for $\gamma = 1$, implying that L_1 is an $L(j,k)$-labeling of $P_n(r)$. Thus $\lambda_{j,k}\big(P_n(r)\big) \leq sp(L_1) = (r-1)j + k < rj$ since $j > k$. So by Lemma 3.11, $\lambda_{j,k}\big(P_n(r)\big) \geq (r-1)j + k$, giving the result.

Case 2. Fix n, $2r+1 \leq n \leq 3r$. Then $\gamma r+1 \leq n \leq (\gamma+1)r$ for $\gamma = 2$, implying that L_2 is an $L(j,k)$-labeling of $P_n(r)$. Thus, for $j > 2k$, $\lambda_{j,k}\big(P_n(r)\big) \leq sp(L_2) = (r-1)j + 2k < rj$, implying by Lemma 3.11 that $\lambda_{j,k}\big(P_n(r)\big) = (r-1)j + 2k$. But this implies that for $x > 2$, $\lambda_{x,1}\big(P_n(r)\big) = (r-1)x + 2$, from which it follows by the continuity of $\lambda_{x,1}(G)$ that $\lambda_{2,1}\big(P_n(r)\big) = 2(r-1) + 2$. Equivalently, we have that $\lambda_{j,k}\big(P_n(r)\big) = (r-1)j + 2k$ for $j = 2k$, giving the result.

Case 3. Fix n, $\gamma r + 1 \leq n \leq (\gamma+1)r$ for $3 \leq \gamma \leq r$. Then L_γ is an $L(j,k)$-labeling of $P_n(r)$ with span $sp(L_\gamma) = (r-1)j + \gamma k \geq \lambda_{j,k}\big(P_n(r)\big)$. If $\frac{j}{k} > \gamma$, then $(r-1)j + \gamma k < rj$, implying by Lemma 3.11 that $\lambda_{j,k}\big(P_n(r)\big) \geq (r-1)j + \gamma k$. Hence $\lambda_{j,k}\big(P_n(r)\big) = (r-1)j + \gamma k$ for $\frac{j}{k} > \gamma$. If $2 \leq \frac{j}{k} < \gamma$, then $rj < (r-1)j + \gamma k$, implying by Lemma 3.2 that $\lambda_{j,k}\big(P_n(r)\big) \leq rj < (r-1)j + \gamma k$. Thus, by Lemma 3.11, $\lambda_{j,k}\big(P_n(r)\big) \geq rj$, giving $\lambda_{j,k}\big(P_n(r)\big) = rj$.

Finally, by the continuity of $\lambda_t\big(P_n(r)\big)$, $\lambda_{j,k}\big(P_n(r)\big) = rj = (r-1)j + \gamma k$ for $\frac{j}{k} = \gamma$.

Case 4. Fix n, $(r+1)r < n \leq \infty$, and suppose that $\frac{j}{k} \geq r$. By Case 3 for $\gamma = r$, $\lambda_{j,k}\big(P_{(r+1)r}(r)\big) = (r-1)j + rk$. But $\lambda_{j,k}\big(P_n(r)\big) \geq \lambda_{j,k}\big(P_{(r+1)r}(r)\big)$ since $P_{(r+1)r}(r)$ is a subgraph of $P_n(r)$. Thus $\lambda_{j,k}\big(P_n(r)\big) \geq (r-1)j + rk$. It now suffices to demonstrate an $L(j,k)$-labeling of $P_n(r)$ with span $(r-1)j + rk$.

Let L^* be an integer assignment to the vertices of $P_n(r)$ such that $L^*(v_{x,y}) = (r-1-y)j + (r-1-y)k + kI_{x\,odd}$, where $I_{x\,odd}$ is 1 if x is odd and 0 otherwise. Then the minimum and maximum labels assigned by L^* are 0 and $(r-1)j + rk$.

If $v_{\alpha,\beta}$ and $v_{\alpha',\beta'}$ are adjacent, then $|L^*(v_{\alpha',\beta'}) - L^*(v_{\alpha,\beta})| = |j(\beta-\beta') + k(\beta-\beta') + k(I_{\alpha' odd} - I_{\alpha odd})| \geq j$ by Observation 3.3. And, if $v_{\alpha,\beta}$ and $v_{\alpha',\beta'}$ are at distance two, then $|L^*(v_{\alpha,\beta}) - L^*(v_{\alpha',\beta'})| = |j(\beta' - \beta) + k(\beta' - \beta) + k(I_{\alpha odd} - I_{\alpha' odd})| \geq k$ by Observation 3.4.

Case 5. Fix n, $(r+1)r < n \leq \infty$, and suppose that $2 \leq \frac{j}{k} < r$. By Case 3 for $\gamma = r$, $\lambda_{j,k}\big(P_{(r+1)r}(r)\big) = rj$. But $\lambda_{j,k}\big(P_n(r)\big) \geq \lambda_{j,k}(P_{(r+1)r}(r))$ since $P_{(r+1)r}(r)$ is a subgraph of $P_n(r)$. Thus $\lambda_{j,k}\big(P_n(r)\big) \geq rj$. But by Lemma 3.2, $\lambda_{j,k}\big(P_n(r)\big) \leq rj$, giving the result. □

We illustrate $\lambda_{j,k}$-labelings of $P_n(5)$ for various n, respectively representing the labeling patterns of L_γ for $\gamma = 3, 4$ and L^* of Case 4.

$$\begin{array}{ccccc} 4j & 3j & 2j & j & 0 \\ 4j+k & 3j+k & 2j+k & j+k & k \\ 4j+2k & 3j+2k & 2j+2k & j+2k & 2k \\ 4j+3k & 3j+3k & & & \end{array}$$

A $\lambda_{j,k}$-labeling of $P_{17}(5)$; $\gamma = 3$, $\dfrac{j}{k} \geq 3$

$$\begin{array}{ccccc} 4j & 3j & 2j & j & 0 \\ 4j+k & 3j+k & 2j+k & j+k & k \\ 4j+2k & 3j+2k & 2j+2k & j+2k & 2k \\ 4j+3k & 3j+3k & 2j+3k & j+3k & 3k \\ 4j+4k & 3j+4k & 2j+4k & j+4k & 4k \end{array}$$

A $\lambda_{j,k}$-labeling of $P_{25}(5)$; $\gamma = 4$, $\dfrac{j}{k} \geq 4$

$$\begin{array}{ccccc} 4j+5k & 3j+4k & 2j+3k & j+2k & k \\ 4j+4k & 3j+3k & 2j+2k & j+k & 0 \\ 4j+5k & 3j+4k & 2j+3k & j+2k & k \\ 4j+4k & 3j+3k & 2j+2k & j+k & 0 \\ 4j+5k & 3j+4k & 2j+3k & j+2k & k \\ 4j+4k & 3j+3k & 2j+2k & j+k & 0 \\ 4j+5k & & & & \end{array}$$

A $\lambda_{j,k}$-labeling of $P_{31}(5)$; $\dfrac{j}{k} \geq 5$.

Corollary 3.12. *For integers j, k such that $\frac{j}{k} \geq 2$,*

$$\lambda_{j,k}\big(P_\infty(r)\big) = \begin{cases} (r-1)j + rk & \text{if } \dfrac{j}{k} > r \\ rj & \text{if } \dfrac{j}{k} \leq r. \end{cases}$$

4. On $\lambda_{j,k}(P_\infty(r))$ for $1 \leq \frac{j}{k} \leq 2$

By Theorems 2.6 and 2.3, we have respectively seen $\lambda_{1,1}\big(P_\infty(r)\big) = 2r-2$ and $\lambda_{j,k}\big(P_\infty(2)\big) = 2j$ for $1 \leq \frac{j}{k} \leq 2$. In this section, we investigate the behavior of $\lambda_{j,k}\big(P_\infty(r)\big)$ (and hence $\lambda_{x,1}\big(P_\infty(r)\big)$) for $r \geq 3$ and $1 \leq \frac{j}{k} \leq 2$. We establish a general upper bound for $\lambda_{j,k}\big(P_\infty(r)\big)$, then determine that this bound is sharp on various subintervals of $[1, 2]$. We also establish $\lambda_{j,k}\big(P_\infty(3)\big)$ and $\lambda_{j,k}\big(P_\infty(4)\big)$ for all $\frac{j}{k}$, $1 \leq \frac{j}{k} \leq 2$

We begin with an upper bound for $\lambda_{j,k}\big(P_\infty(r)\big)$.

Theorem 4.1. *For $r \geq 3$ and $1 \leq \frac{j}{k} \leq 2$,*

$$\lambda_{j,k}\big(P_\infty(r)\big) \leq \begin{cases} (2r-2)j & \text{if } 1 \leq \dfrac{j}{k} \leq \dfrac{r}{r-1} \\ 2rk & \text{if } \dfrac{r}{r-1} \leq \dfrac{j}{k} \leq 2. \end{cases}$$

Proof. Let L_1 assign integers to the vertices of $P_\infty(r)$ according to the repeated pattern

$$0, 2k, 4k, \ldots, 2rk, k, 3k, 5k, \ldots, (2r-1)k$$

where $L_1(v_0) = 0$, $L_1(v_1) = 2k$, and so on. Let L_2 assign integers to the vertices of $P_\infty(r)$ such that $L_2(v_i) = j(i \bmod (2r-1))$. It is easily checked that for $1 \leq \frac{j}{k} \leq 2$, L_1 and L_2 are $L(j,k)$-labelings of $P_\infty(r)$ with respective spans $2rk$ and $(2r-2)j$. Thus for fixed $\frac{j_0}{k_0} \in [1, 2]$, $\lambda_{j_0,k_0}\big(P_\infty(r)\big) \leq \min\{2rk_0, (2r-2)j_0\}$. The result follows. $\square$

We are now able to quickly characterize $\lambda_{j,k}\big(P_\infty(r)\big)$ for $1 \leq \frac{j}{k} \leq \frac{2r-2}{2r-3}$.

Theorem 4.2. *For $r \geq 3$ and $1 \leq \frac{j}{k} \leq \frac{2r-2}{2r-3}$, $\lambda_{j,k}\big(P_\infty(r)\big) = (2r-2)j$.*

Proof. Noting that $P_\infty(r)$ is a $(2r-2)$-regular graph and noting that every $L(j,k)$-labeling of $P_\infty(r)$ induces an $L(j,k)$-labeling of the infinite $(2r-2)$-regular tree $T_\infty(2r-2)$, we conclude that $\lambda_{j,k}\big(T_\infty(2r-2)\big) \leq \lambda_{j,k}\big(P_\infty(r)\big)$. But by [4], $\lambda_{j,k}\big(T_\infty(2r-2)\big) = (2r-2)j$ for $1 \leq \frac{j}{k} \leq \frac{2r-2}{2r-3}$. The result now follows from Theorem 4.1. $\square$

We next determine $\lambda_{j,k}\big(P_\infty(r)\big)$ for $\frac{3}{2} \leq \frac{j}{k} \leq 2$, beginning with two lemmas that will be helpful in this case and also in the consideration of $r = 3, 4$.

For graph G and non-negative integers i, j, and k such that $1 \leq k \leq j$, let $T_{j,i}$ denote the set of integers strictly between ij and $(i+1)j$, and let $\Lambda_{j,k}(G)$ denote the set of normalized $\lambda_{j,k}$-labelings of G; that is, the set of all $\lambda_{j,k}$-labelings of G such that each assigns labels only of the form $mj + bk$ for some non-negative integers m, b. Let $\Lambda^*_{j,k}(G)$ be the collection of all labelings in $\Lambda_{j,k}(G)$ that assign at least one label not divisible by j. Then clearly every labeling in $\Lambda^*_{j,k}(G)$ assigns at least one label from $\bigcup_{i=0}^{z} T_{j,i}$, where z denotes $\lfloor \frac{\lambda_{j,k}(G)}{j} \rfloor$.

Lemma 4.3. *Let $1 \leq k < j$ and let G be a graph such that no labeling in $\Lambda_{j,k}(G)$ assigns k to some vertex of G. Then for every non-negative integer p, no labeling in $\Lambda_{j,k}(G)$ assigns the label $pj + k$ to a vertex of G.*

Proof. Assuming otherwise, we may find the smallest non-negative integer p_0 such that for some L in $\Lambda_{j,k}(G)$ and some vertex v^* of G, $L(v^*) = p_0 j + k$. We note that $p_0 \neq 0$ by the lemma's hypothesis that k is assigned to no vertex under L. We also note that if L assigns some label from $T_{j,0} =]0, j[$ to some vertex w of G, then (due to the normality of L), that label cannot be in the interval $]0, k[$, and is hence a multiple of k in $]k, j[$. We may thus form the following normalized $\lambda_{j,k}$-labeling L_1 of G that assigns k to w, contradicting the hypotheses of the lemma:

$$L_1(v) = \begin{cases} k & \text{if } v = w \\ L(v) & \text{if } v \neq w. \end{cases}$$

Therefore $T_{j,0}$ contains no labels assigned by L, implying that the two smallest labels assigned by L are 0 and j. We may hence form the following normalized $\lambda_{j,k}$-labeling L_2 of G such that $L_2(v^*) = (p_0 - 1)j + k$, a contradiction of either the minimality of p_0 (if $p_0 \geq 2$) or the assumption that no normalized $\lambda_{j,k}$-labeling of G assigns the label k (if $p_0 = 1$):

$$L_2(v) = \begin{cases} L(v) - j & \text{if } L(v) \neq 0 \\ \lambda_{j,k}(G) & \text{if } L(v) = 0. \end{cases}$$

Thus for each non-negative integer p, no labeling in $\Lambda_{j,k}(G)$ assigns $pj + k$ to some vertex of G. $\square$

Lemma 4.4. *Let $1 \leq k \leq j$ and let G be a graph such that no labeling in $\Lambda_{j,k}(G)$ assigns k to some vertex of G. Then every labeling in $\Lambda_{j,k}(G)$ assigns only multiples of j to the vertices of G, and $\lambda_{j,k}(G) = j\lambda_{1,1}(G)$.*

Proof. If G has no edges or $j = k$, then the claim is obviously true. Thus, we assume that $j > k$ and G has at least one edge, implying $\lambda_{j,k}(G) \geq j$.

We now proceed by contradiction to show that $\Lambda^*_{j,k}(G)$ is empty. Assuming the contrary, we can choose $L^* \in \Lambda^*_{j,k}(G)$ such that the number $m \geq 1$ of sets $T_{j,i}$ containing labels assigned by L^* is minimized.

We consider two cases.

Case 1. $m \geq 2$. Let p_0 be the smallest integer such that T_{j,p_0} contains a label assigned by L^*. Then since $m \geq 2$, the following labeling L_3 is easily seen to be in $\Lambda^*_{j,k}(G)$:

$$L_3(v) = \begin{cases} p_0 j & \text{if } p_0 j < L^*(v) < p_0 j + k \\ L^*(v) & \text{otherwise.} \end{cases}$$

Since L_3 assigns no label from the interval $]p_0 j, p_0 j + k[$, the minimality of m and Lemma 4.3 imply that L_3 assigns at least one label from the interval $]p_0 j + k, (p_0 + 1)j[$. Let v^* be a vertex that receives the smallest label under L_3 from among assigned labels in $]p_0 j + k, (p_0 + 1)j[$. We may form the following labeling $L_4 \in \Lambda^*_{j,k}(G)$ that assigns label $p_0 j + k$ to v^*, contradicting Lemma 4.3:

$$L_4(v) = \begin{cases} p_0 j + k & \text{if } v = v^* \\ L_3(v) & \text{if } v \neq v^*. \end{cases}$$

Thus $m = 1$.

Case 2. $m = 1$. We assume that among the labelings of $\Lambda^*_{j,k}(G)$ using the labels of only one $T_{j,i}$, the labeling L^* assigns the fewest number y of distinct labels not divisible by j. Let $T_{j,p}$ be the set containing these labels. We first argue that $y = 1$, then argue that $y \neq 1$.

Suppose $y \geq 2$. If L^* assigns a label from the interval $]pj, pj+k[$ to some vertex v^* in G, then we form a new labeling $L_5 \in \Lambda^*_{j,k}(G)$ that assigns $y-1$ distinct labels from $T_{j,p}$:

$$L_5(v) = \begin{cases} pj & \text{if } L^*(v) = L^*(v^*) \\ L^*(v) & \text{if } L^*(v) \neq L^*(v^*). \end{cases}$$

Since this contradicts the minimality of y, then by Lemma 4.3 the y distinct labels assigned by L^* from $T_{j,p}$ are in $]pj+k, (p+1)j[$. Now let v^* be a vertex that receives the smallest label from among labels assigned by L^* in the interval $]pj+k, (p+1)j[$. We may form a labeling $L_6 \in \Lambda^*_{j,k}(G)$ that assigns label $pj+k$ to v^*, contradicting Lemma 4.3:

$$L_6(v) = \begin{cases} pj+k & \text{if } v = v^* \\ L^*(v) & \text{if } v \neq v^*. \end{cases}$$

We may thus assume that $y = 1$. If L^* assigns a label from the interval $]pj+k, (p+1)j[$ to some vertex v^*, then we form L_6 above, arriving at the indicated contradiction. Hence L^* assigns to some vertex v^* a label in $]pj, pj+k]$ (and thus, by Lemma 4.3, a label in $]pj, pj+k[$). We observe that $p \neq z$, since otherwise $\lambda_{j,k}(G) = L^*(v^*)$ and the labeling L_7 would be an $L(j,k)$-labeling with span smaller than $L^*(v^*)$:

$$L_7(v) = \begin{cases} pj & \text{if } L^*(v) = L^*(v^*) \\ L^*(v) & \text{if } L^*(v) \neq L^*(v^*). \end{cases}$$

This implies that $0 \leq p < z$ and $\lambda_{j,k}(G) = zj$. We are now able to show that L^* leads to a normalized $\lambda_{j,k}$-labeling of G that assigns the label k, a contradiction. Consider the labeling L_8 as follows:

$$L_8(v) = \begin{cases} pj+1 & \text{if } L^*(v) = L^*(v^*) \\ L^*(v) & \text{if } L^*(v) \neq L^*(v^*). \end{cases}$$

It is clear that this is a $\lambda_{j,k}$-labeling of G (though not necessarily a normalized one). It is also clear that the following labeling L_9 is a (not necessarily normalized) $\lambda_{j,k}$-labeling of G such that the smallest assigned label is 0 and $j-1$ is assigned to the vertices that receive the label $pj+1$ under L_8:

$$L_9(v) = \begin{cases} (p+1)j - L_8(v) & \text{if } L_8(v) \leq (p+1)j \\ L_8(v) & \text{if } L_8(v) > (p+1)j. \end{cases}$$

Since under L_9 the only label assigned from the interval $]0, j[$ is $j-1$, we form the following normalized labeling that assigns the label k:

$$L_{10}(v) = \begin{cases} k & \text{if } L_9(v) = j-1 \\ L_9(v) & \text{if } L_9(v) \neq j-1. \end{cases}$$

But this contradicts the assumption that no labeling in $\Lambda_{j,k}(G)$ assigns k. Hence we have $y \neq 1$, and thus $m \neq 1$, concluding the proof. $\square$

We are now able to establish $\lambda_{j,k}\big(P_\infty(r)\big)$ for $\frac{3}{2} \leq \frac{j}{k} \leq 2$.

Theorem 4.5. *For $r \geq 3$ and $\frac{3}{2} \leq \frac{j}{k} \leq 2$, $\lambda_{j,k}\big(P_\infty\big) = 2rk$.*

Proof. We first show that $\lambda_{3,2}\big(P_\infty(r)\big) = 4r$. By Theorem 4.1, $\lambda_{3,2}\big(P_\infty(r)\big) \leq 4r$. Thus suppose to the contrary that $\lambda_{3,2}\big(P_\infty(r)\big)$ is strictly less than $4r$. If no $\lambda_{3,2}$-labeling of $P_\infty(r)$ assigns the label 2, then by Theorem 4.2 and Lemma 4.4, $\lambda_{3,2}\big(P_\infty(r)\big) = 3\lambda_{1,1}\big(P_\infty(r)\big) = 3(2r-2) \geq 4r$, a contradiction. So we may select normalized $\lambda_{3,2}$-labeling L of $P_\infty(r)$ such that with no loss of generality, $L(v_0) = 2$. Then due to the fact that the $2r-2$ neighbors of v_0 are pairwise at most two apart, the largest label assigned to those neighbors is at least $4r-1$. Thus $\lambda_{3,2}\big(P_\infty(r)\big) = 4r-1$, implying that the neighbors of v_0 must be assigned the odd labels $5, 7, 9, 11, \ldots, 4r-1$ by L. Now suppose w is the neighbor of v_0 such that $L(w) = 4r-3$. Then the neighbors of w must receive the labels of $0, 2, 4, , \ldots, 4r-6$ or else the span of L is violated. Therefore, since w and v_0 are adjacent, they have at least $r-2 \geq 1$ common neighbors with labels that are both odd and even, a contradiction. Hence, $\lambda_{3,2}\big(P_\infty(r)\big) = 4r$.

We are now able to establish $\lambda_{j,k}\big(P_\infty(r)\big)$ for $\frac{3}{2} < \frac{j}{k} \leq 2$. Since $\lambda_{3,2}\big(P_\infty(r)\big) = 4r$, then $\lambda_{\frac{3}{2},1}\big(P_\infty(r)\big) = 2r$. But by Theorem 3.1, $\lambda_{2,1}\big(P_\infty(r)\big) = 2r$ as well. Therefore, by the monotonicity and continuity properties of $\lambda_{x,1}\big(P_\infty(r)\big)$ given by Theorem 2.1, we have $\lambda_{x,1}\big(P_\infty(r)\big) = 2r$ for $\frac{3}{2} < x \leq 2$. Hence, for $\frac{3}{2} \leq \frac{j}{k} \leq 2$, $\lambda_{\frac{j}{k},1}\big(P_\infty(r)\big) = 2r$, implying $\lambda_{j,k}\big(P_\infty(r)\big) = 2rk$. $\square$

By Theorems 4.2 and 4.5, we now have the behavior of $\lambda_{j,k}\big(P_\infty(3)\big)$ for $\frac{j}{k}$ in the intervals $[1, \frac{4}{3}]$ and $[\frac{3}{2}, 2]$. Our next result gives the behavior of $\lambda_{j,k}\big(P_\infty(3)\big)$ throughout the entire interval $[1, 2]$.

Theorem 4.6. *For* $1 \le \frac{j}{k} \le 2$,

$$\lambda_{j,k}\big(P_\infty(3)\big) = \begin{cases} 4j & \text{if } 1 \le \dfrac{j}{k} \le \dfrac{3}{2} \\ 6k & \text{if } \dfrac{3}{2} \le \dfrac{j}{k} \le 2. \end{cases}$$

Proof. It suffices to determine the behavior of $\lambda_{j,k}\big(P_\infty(3)\big)$ for $\frac{j}{k} \in [\frac{4}{3}, \frac{3}{2}]$.

We first note that $\lambda_{\frac{4}{3},1}\big(P_\infty(3)\big) = \frac{16}{3}$ by Theorem 4.2 and $\lambda_{\frac{3}{2},1}\big(P_\infty(3)\big) = 6$ by Theorem 4.5. Thus, by Theorem 2.1, $\lambda_{x,1}\big(P_\infty(3)\big)$ is non-decreasing piecewise linear, passing through $(\frac{4}{3}, \frac{16}{3})$ and $(\frac{3}{2}, 6)$, two points connected by the line $y = 4x$. If $\lambda_{x,1}\big(P_\infty(3)\big)$ is not $4x$ on the entire interval $[\frac{4}{3}, \frac{3}{2}]$, then on some interval $[x_1, x_2] \subset [\frac{4}{3}, \frac{3}{2}]$, $\lambda_{x,1}\big(P_\infty(3)\big) = mx + b$ where $m > 4$ and $b < 0$, a contradiction of the non-negativity of b as guaranteed by Theorem 2.1. Thus $\lambda_{x,1}\big(P_\infty(3)\big) = 4x$ on $[\frac{4}{3}, \frac{3}{2}]$, giving $\lambda_{j,k}\big(P_\infty(3)\big) = 4j$ on $[\frac{4}{3}, \frac{3}{2}]$. $\square$

In next turning our attention to the derivation of $\lambda_{j,k}\big(P_\infty(4)\big)$, we note that Theorems 4.2 and 4.5 give us $\lambda_{j,k}\big(P_\infty(4)\big) = 6j$ for $1 \le \frac{j}{k} \le \frac{6}{5}$ and $\lambda_{j,k}\big(P_\infty(4)\big) = 8k$ for $\frac{3}{2} \le \frac{j}{k} \le 2$. However, the latter result is not sufficiently refined for an immediate derivation of $\lambda_{j,k}\big(P_\infty(4)\big)$ on the entire interval $[0, 1]$. Thus, we state and prove Theorem 4.7, analogous to Theorem 4.5, from which the characterization of $\lambda_{j,k}\big(P_\infty(4)\big)$ will follow. Due to the length of the proof of Theorem 4.7, we postpone it until the section's end.

Theorem 4.7. *For* $r \ge 4$ *and* $\frac{4}{3} \le \frac{j}{k} \le 2$, $\lambda_{j,k}\big(P_\infty\big) = 2rk$.

Proof. Deferred.

Theorem 4.8. *For* $1 \le \frac{j}{k} \le 2$,

$$\lambda_{j,k}\big(P_\infty(4)\big) = \begin{cases} 6j & \text{if } 1 \le \dfrac{j}{k} \le \dfrac{4}{3} \\ 8k & \text{if } \dfrac{4}{3} \le \dfrac{j}{k} \le 2. \end{cases}$$

Proof. By Theorems 4.2 and 4.7, it suffices to determine the behavior of $\lambda_{j,k}\big(P_\infty(4)\big)$ for $\frac{j}{k} \in [\frac{6}{5}, \frac{4}{3}]$.

We first note that $\lambda_{\frac{6}{5},1}\big(P_\infty(4)\big) = \frac{36}{5}$ by Theorem 4.2 and $\lambda_{\frac{4}{3},1}\big(P_\infty(4)\big) = 8$ by Theorem 4.5. Thus, by Theorem 2.1, $\lambda_{x,1}\big(P_\infty(4)\big)$ is non-decreasing and piecewise linear, passing through $(\frac{6}{5}, \frac{36}{5})$ and $(\frac{4}{3}, 8)$, two points joined by the line $y = 6x$. If $\lambda_{x,1}\big(P_\infty(4)\big)$ is not $6x$ on the entire interval $[\frac{6}{5}, \frac{4}{3}]$, then on some interval $[x_1, x_2] \subset [\frac{6}{5}, \frac{4}{3}]$, $\lambda_{x,1}\big(P_\infty(4)\big) = mx + b$ where $m > 4$ and $b < 0$, a contradiction of the non-negativity of b as guaranteed by Theorem 2.1. Thus $\lambda_{x,1}\big(P_\infty(4)\big) = 6x$ on $[\frac{6}{5}, \frac{4}{3}]$, giving $\lambda_{j,k}\big(P_\infty(4)\big) = 6j$ on $[\frac{6}{5}, \frac{4}{3}]$. $\square$

We now turn to some definitions and observations in support of the proof of Theorem 4.7.

Let x and y be integers, $x<y$. Throughout the remainder of the section, we let $V_{x,y}$ denote the subset $\{v_x, v_{x+1}, x_{x+2}, \ldots, v_y\}$ of $V\big(P_\infty(r)\big)$.

Lemma 4.9. *Suppose that no $\lambda_{j,k}$-labeling L of $P_\infty(r)$ assigns k to some vertex. Then* $\lambda_{j,k}\big(P_\infty(r)\big) \ge (2r-2)j$.

Proof. The hypothesis implies that no normalized $\lambda_{j,k}$-labeling L of $P_\infty(r)$ assigns k to some vertex. By Lemma 4.4, L assigns only non-negative multiples of j to the vertices of $P_\infty(r)$. Since the subgraph of $P_\infty(r)$ induced by $V_{1,2r-1}$ has diameter 2, the labels assigned to those vertices by L are distinct, implying that the largest assigned label is at least $(2r-2)j$. $\square$

Lemma 4.10. *For $r \ge 4$, let L be a $\lambda_{4,3}$-labeling of $P_\infty(r)$ such that $sp(L) = 6r - 1$, $L(v_0) = 3$, and the largest label assigned among vertices in $V_{1-r,r-1}$ is $6r - 1$. Then the labels assigned to those vertices form a strictly increasing sequence $s = \langle a_0, a_1, a_2, \ldots, a_{2r-2}\rangle$ such that*

(i) $a_0 = 3$;
(ii) $a_1 = 7$;
(iii) *there exists a unique i_0, $1 \le i_0 \le 2r-3$, such that $a_{i_0+1} - a_{i_0} = 4$, and*
(iv) *for all $i \ne i_0$, $a_{i+1} - a_i = 3$.*

Proof. We first note that v_0 is adjacent to every vertex in $V_{1-r,r-1} - \{v_0\}$. Thus by the distance one condition no integer smaller than 3 is assigned by L to vertices in $V_{1-r,r-1}$, establishing part (i).

We next observe that the vertices in $V_{1-r,r-1}$ are pairwise adjacent or distance two apart, implying that the labels of any two distinct vertices differ by at least 3. The labels assigned to the vertices in $V_{1-r,r-1}$ hence form a strictly increasing sequence $\langle 3, a_1, a_2, \ldots, a_{2r-2}\rangle$ where $a_{i+1} - a_i \geq 3$ for $1 \leq i \leq 2r-3$ and in particular, $a_1 \geq 7$ by the distance one condition and the hypothesis $L(v_0) = 3$. Since we have also hypothesized that $a_{2r-2} = 6r - 1$, it follows that $6r - 1 \geq a_1 + 3(2r-3)$, giving $a_1 \leq 8$. We show that a_1 cannot be 8.

Suppose to the contrary that $a_1 = 8$. Since $a_{2r-2} = 6r - 1$, then $a_{i+1} - a_i = 3$ for $1 \leq i \leq 2r - 3$, implying that the labels assigned to the vertices in $V_{1-r,r-1}$ form the strictly increasing sequence $\langle 3, 8, 11, \ldots, 6r-7, 6r-4, 6r-1\rangle$. Since all of the vertices in $V_{1,r-1}$ (resp. $V_{1-r,-1}$) are pairwise adjacent, we may assume without loss of generality that the labels of the vertices in $V_{1,r-1}$ form the strictly increasing sequence $s_1 = \langle 8, 14, 20, \ldots, 6r-10, 6r-4\rangle$, and the labels of the vertices in $V_{1-r,-1}$ form the strictly increasing sequence $s_2 = \langle 11, 17, 23, 29, \ldots, 6r-7, 6r-1\rangle$. Noting that v_1 is adjacent to every vertex in $V_{1-r,-1}$ except v_{1-r}, it follows that $L(v_1) = 8$. Now consider v_r, a vertex that is adjacent to all vertices in $V_{1,r-1}$ receiving labels from the sequence s_1 under L and at distance two from v_0 of label 3. Then $L(v_r) \leq 3 - 3 = 0$ or $L(v_r) \geq (6r-4) + 4 = 6r$. Since the latter case is not possible by the hypothesis $sp(L) = 6r - 1$, we see that $L(v_r) = 0$. Note that v_{r+1} is adjacent to vertices of labels $0, 14, 20, \ldots, 6r-10, 6r-4$ and at distance two from the vertices v_0, v_1 of respective labels 3 and 8, implying $L(v_{r+1}) \geq 6r$, a contradiction. Hence $a_1 \neq 8$, implying $a_1 = 7$, establishing part (ii). Parts (iii) and (iv) easily follow from our assumption that $a_{2r-2} = 6r - 1$. □

Lemma 4.11. *For $r \geq 4$, let L be a $\lambda_{4,3}$-labeling of $P_\infty(r)$ such that $L(v_0) = 3$, $sp(L) = 6r - 1$, and the largest label assigned among vertices in $V_{1-r,r-1}$ is $6r - 1$. Let $s = \langle a_1, a_2, \ldots, a_{2r-2}\rangle$ be the strictly increasing sequence of labels assigned by L to the vertices in $V_{1-r,r-1} - \{v_0\}$ that is guaranteed to exist by Lemma 4.10. Then $L(v_1)$ and $L(v_{-1})$ are among $a_1 = 7$, a_{i_0}, a_{i_0+1}, and $a_{2r-2} = 6r - 1$.*

Proof. Suppose that $L(v_1) = a_m$ where a_m is not among a_1, a_{i_0}, a_{i_0+1} and a_{2r-2}. Since $2 \leq m \leq 2r - 3$ and since v_1 is adjacent to every vertex in $V_{1-r,r-1} - \{v_0\}$ except v_{1-r}, then either $a_{m+1} - a_m \geq 4$ or $a_m - a_{m-1} \geq 4$, each of which contradicts properties of s established in Lemma 4.10. A similar contradiction is obtained if $L(v_{-1}) = a_m$ where a_m is not among a_1, a_{i_0}, a_{i_0+1} and a_{2r-2}. □

Proof of Theorem 4.7. By Theorem 3.1, $\lambda_{2,1}(P_\infty(r)) = 2r$. Thus, if we can show that $\lambda_{4,3}(P_\infty(r)) = 6r$, we will then have that $\lambda_{\frac{4}{3},1}(P_\infty(r)) = 2r$, implying that $\lambda_{x,1}(P_\infty(r)) = 2r$ for $\frac{4}{3} \leq x \leq 2$ by Theorem 2.1, which in turn will give the desired result $\lambda_{j,k}(P_\infty(r)) = 2rk$ for $\frac{4}{3} \leq \frac{j}{k} \leq 2$. We thus devote the rest of the proof to showing that $\lambda_{4,3}(P_\infty(r)) = 6r$.

Let L denote a $\lambda_{4,3}$-labeling of $P_\infty(r)$.

Claim 1. $sp(L) \geq 6r - 1$.

Assume to the contrary that $sp(L) \leq 6r - 2$. If no $\lambda_{4,3}$-labeling of $P_\infty(r)$ assigns label $k = 3$ to some vertex, then by Lemma 4.9 and the hypothesis $r \geq 4$, $sp(L) \geq 4(2r-2) = 8r - 8 \geq 6r$. Thus we may assume (with no loss of generality) that L assigns the label 3 to v_0. Now consider the subgraph of $P_\infty(r)$ induced by $V_{1-r,r-1}$. Since this subgraph has diameter 2 and v_0 is adjacent to every vertex in $V_{1-r,r-1} - \{v_0\}$, then the labels assigned to the vertices of $V_{1-r,r-1}$ necessarily form the strictly increasing sequence $\langle 3, 7, 10, 13, \ldots, 6r-2\rangle$. Now suppose w is the neighbor of v_0 such that $L(w) = 6r - 5$. Then the neighbors of w must receive the labels of $0, 3, 6, 9, \ldots, 6r-9$ or else the span of L is violated. Since w and v_0 are adjacent, they have at least $r - 2 \geq 2$ common neighbors with labels that are both multiples of 3 and not multiples of 3, a contradiction. Claim 1 is thus demonstrated.

But since Theorem 4.1 tells us that $\lambda_{4,3}(P_\infty(r)) \leq 6r$, it follows that $\lambda_{4,3}(P_\infty(r))$ is either $6r - 1$ or $6r$. Therefore, we assume to the contrary that L is a $\lambda_{4,3}$-labeling of $P_\infty(r)$ with span $6r - 1$. As above, we may further assume that L assigns the label $k = 3$ to vertex v_0. Then by the distance conditions with $j = 4$ and $k = 3$, the largest label assigned by L among the vertices in $V_{1-r,r-1}$ is at least $7 + (2r-3)k = 6r - 2$ (and at most $6r - 1$ by the hypothesized span of L). However, since any vertex with label $6r - 2$ may be relabeled $6r - 1$ without causing a violation of the distance conditions, we may assume that the largest label assigned by L among the vertices in $V_{1-r,r-1}$ is $6r - 1$.

By Lemma 4.10, let $s = \langle a_0, a_1, a_2, a_3, \ldots, a_{2r-2}\rangle$ denote the strictly increasing sequence of distinct labels assigned by L to the vertices in $V_{1-r,r-1}$, and let i_0 be the unique integer such that $a_{i_0+1} - a_{i_0} = 4$, $1 \leq i_0 \leq 2r - 3$. Then

$$a_i = \begin{cases} 3 & \text{if } i = 0 \\ 3i + 4 & \text{if } 1 \leq i \leq i_0 \\ 3i + 5 & \text{if } i_0 + 1 \leq i \leq 2r - 2. \end{cases}$$

It follows that either $a_{2r-3} = 6r - 4$ or $a_{2r-3} = 6r - 5$. In the latter case, $i_0 = 2r - 3$ and $s = \langle 7, 10, 13, \ldots, 6r-5, 6r-1\rangle$. So by the distance one condition, with no loss of generality, we may assume that the labels of the vertices in $V_{1,r-1}$ form the strictly increasing sequence $s_1 = \langle 7, 13, 19, \ldots, 6r-5\rangle$. By Lemma 4.11, we have $L(v_{-1}) = 6r - 1$. But it is easily verified that the distance conditions then imply $L(v_r) = 0$, leaving no label available for v_{r+1}. Thus $a_{2r-3} = 6r - 4$ and $i_0 \leq 2r - 4$.

Claim 2. *Either* $i_0 = 2r-5$ *or* $i_0 = 2r-4$.

By contradiction, assume that $i_0 \le 2r-6$, implying that $a_{2r-5} = 6r-10$ and $a_{2r-3} = 6r-4$. Since the vertices with labels a_{2r-3} and a_{2r-5} cannot be in different sets $V_{1,r-1}$ and $V_{1-r,-1}$ without violating the distance one condition with respect to the vertex with label $a_{2r-4} = 6r-7$, we may assume with no loss of generality that the vertices with labels a_{2r-5} and a_{2r-3} are in $V_{1,r-1}$ and hence adjacent. But $(6r-1)-L$ is also a $\lambda_{4,3}$-labeling of $P_\infty(r)$. Hence, the two adjacent vertices of respective labels $6r-4$ and $6r-10$ under L will receive respective labels 3 and 9 under the labeling $(6r-1)-L$, contradicting Lemma 4.10. Therefore Claim 2 is demonstrated.

By the preceding discussion, we now have $sp(L) = 6r-1$, $a_{2r-3} = 6r-4$, and $i_0 = 2r-5$ or $2r-4$. The remainder of the proof will be structured on the cases that, with no loss of generality, either both vertices with labels a_{i_0} and a_{i_0+1} are in $V_{1,r-1}$ or only the vertex with label a_{i_0} is in $V_{1,r-1}$.

Case 1: The set $V_{1,r-1}$ contains vertices with labels a_{i_0} and a_{i_0+1}.

If $i_0 = 2r-5$, then the labels of the vertices in $V_{1-r,r-1} - \{v_0\}$ form the strictly increasing sequence of length $2r-2$ given by $\langle 7, 10, \ldots, 6r-14, 6r-11, 6r-7, 6r-4, 6r-1\rangle$. By inspection, it is impossible to form two disjoint subsequences of length $r-1$ such that one subsequence contains both $6r-11$ and $6r-7$ and each subsequence has elements that differ pairwise by at least 4. Thus, the vertices of $V_{1-r,-1}$ and $V_{1,r-1}$ cannot be properly labeled, and hence it follows that $i_0 = 2r-4$. Under this condition, we may assume with no loss of generality that the labels of the vertices in $V_{1,r-1}$ form the sequence $s_2 = \left\langle 10, 16, \ldots, 6r-8, 6r-4\right\rangle$.

Claim 3. $L(v_1) = 6r-8$.

By Lemma 4.11, either $L(v_1) = 6r-4$ or $L(v_1) = 6r-8$. Suppose $L(v_1) = 6r-4$. Then $L(v_{1-r}) = 6r-1$ because v_{1-r} is the only vertex in $V_{1-r,r-1}$ being at distance two from the vertex v_1. It thus follows from Lemma 4.11 that $L(v_{-1}) = 7$, which in turn implies that $L(v_{r-1}) = 10$, $L(v_r) = 0$ and $L(v_{r+1}) = 6r-1$. Let $L(v_2) = a_k$. Then either $k = i_0 = 2r-4$ or $k < 2r-4$. Noting that v_2 is adjacent to every vertex in $V_{1-r,r-1}$ except v_{1-r} and v_{2-r}, the latter implies that $\{L(v_{1-r}), L(v_{2-r})\} = \{a_{k-1}, a_{k+1}\}$, contradicting $L(v_{1-r}) = 6r-1 = a_{2r-2}$. Thus $L(v_2) = a_{2r-4} = 6r-8$. If $r = 4$, it is easily verified that the labels of v_0, v_1, v_2, v_3, v_4, and v_5 are respectively 3, 20, 16, 10, 0, and 23, implying that the respective labels of v_6, v_7 and v_8 must be 6, 13 and 19, leaving no label available for v_9. Thus we assume $r \ge 5$. Then v_{r+2} is adjacent to vertices with labels $0, 10, 16, \ldots, 6r-14, 6r-1$ and at distance two from the vertices v_{-1}, v_0, v_1, v_2 with respective labels $7, 3, 6r-4, 6r-8$, leaving no label for v_{r+2}, a contradiction. Thus Claim 3 is demonstrated.

With $L(v_1) = 6r-8$, it follows from the span of L and the distance conditions that $L(v_r) \le 6$ and $L(v_{r+1}) \le 6$. Since the two adjacent vertices v_r and v_{r+1} are distance two from the vertex v_0 of label 3, one of $L(v_r)$ and $L(v_{r+1})$ must be 6, implying either $L(v_{1-r}) = 7$ or $L(v_{2-r}) = 7$. So from Lemma 4.11, $L(v_{-1}) = a_{2r-2} = 6r-1$ and hence $L(v_{r-1}) = a_{2r-3} = 6r-4$. Therefore $L(v_2) \ne 6r-4$, implying $L(v_2) = 10 + 6i$ for some i with $0 \le i \le r-4$. Since v_2 is adjacent to every vertex in $V_{1-r,r-1}$ except v_{1-r} and v_{2-r}, we have $\{L(v_{1-r}), L(v_{2-r})\} = \{7+6i, 13+6i\}$. But $i = 0$ because one of $L(v_{1-r})$ and $L(v_{2-r})$ is 7. Thus $L(v_2) = 10$ and $\{L(v_{1-r}), L(v_{2-r})\} = \{13, 7\}$. Since $L(v_1) = 6r-8$ and v_1 is adjacent to every vertex in $V_{1-r,-1}$ except v_{1-r}, then the label $6r-11$ must be assigned to v_{1-r}, implying $6r-11 = 7$ or $6r-11 = 13$. But $r \ge 4$, giving $6r-11 = 13$, which in turn implies $r = 4$. Thus $v_{-3}, v_{-2}, v_{-1}, v_0, v_1$ and v_2 have respective labels 13, 7, 23, 3, 16 and 10. It is easily verified that the distance conditions then require $L(v_{-4}) = 0$ or 19. If $L(v_{-4}) = 19$, then necessarily $L(v_{-5}) = 0$, leaving no label available for v_{-6}. On the other hand, if $L(v_{-4}) = 0$, then either $L(v_{-5}) = 19$ or 20. In each case, no label is available for v_{-6}. Hence Case 1 fails.

Case 2: The set $V_{1,r-1}$ contains precisely one vertex with label a_{i_0} or a_{i_0+1}.

With no loss of generality, let $V_{1,r-1}$ contain a vertex with label a_{i_0}. We know that i_0 is either $2r-5$ or $2r-4$. We finish the proof by demonstrating a contradiction in each case.

Subcase a: $i_0 = 2r-5$.

Then the labels of the vertices in $V_{1,r-1}$ form the strictly increasing sequence

$$s_3 = \left\langle 7, 13, 19, \ldots, 6r-11, 6r-4\right\rangle.$$

By Lemma 4.11, either $L(v_1) = 7$ or $L(v_1) = 6r-11$. The former case leads to no label for v_{r+1}. Thus $L(v_1) = 6r-11$, from which it follows that $L(v_{r+1}) = 6r-8$. This implies that $L(v_r) = 0$. By Lemma 4.11, either $L(v_{-1}) = 6r-7$ or $L(v_{-1}) = 6r-1$. Since v_{-1} is at distance two from v_{r+1} and $L(v_{r+1}) = 6r-8$, we have $L(v_{-1}) = 6r-1$, which in turn implies that $L(v_{r-1}) = 6r-4$. So the labels assigned to the vertices in $V_{1,r-2}$ form the increasing sequence

$$s_4 = \langle a_1, a_3, a_5, \ldots, a_{i_0}\rangle$$

and the labels assigned to the vertices in $V_{1-r,-2}$ form the increasing sequence

$$s_5 = \langle a_2, a_4, a_6, \ldots, a_{i_0+1}\rangle.$$

Since v_{-2} is adjacent to every vertex in $V_{1,r-2}$ except v_{r-2}, the label assigned to v_{-2} from s_5 cannot differ from 2 distinct components of s_4 by exactly 3 without causing a violation of the distance one condition. This implies $L(v_{-2}) = a_{i_0+1} = 6r-7$. But if $r \ge 5$, then the distance two condition is violated since $L(v_{r+1}) = 6r-8$. And if $r = 4$, then it is easy to check that the vertices $v_0, v_1, v_2, v_3, v_4, v_5$ have respective labels 3, 13, 7, 20, 0 and 16, leaving no label available for v_9. Hence $i_0 \ne 2r-5$.

Subcase b: $i_0 = 2r - 4$.

We now have that the labels of the vertices in $V_{1-r,-1}$ form the strictly increasing sequence

$$s_6 = \langle 7, 13, 19, \ldots, 6r - 11, 6r - 4\rangle.$$

By Lemma 4.11, either $L(v_{-1}) = 7$ or $L(v_{-1}) = 6r - 4$. The former case leads to no label for v_{-1-r}. Thus $L(v_{-1}) = 6r - 4$. This implies $L(v_{-r}) = 0$. By Lemma 4.11, $L(v_1) = a_{i_0} = 6r - 8$. Moreover, it follows that $L(v_{-1-r}) = 6r - 1$. But if $r \geq 5$, no label is available for v_{-2-r} by noting that v_{-2-r} is at distance two from v_1 of label $6r - 8$. And if $r = 4$, the labels of $v_{-2}, v_{-1}, v_0, v_1, v_2$, and v_3 are respectively 7, 20, 3, 16, 10, and 23. From the distance conditions, it is then easily verified that $L(v_4)$ and $L(v_5)$ are 0 and 6, respectively. Thus, $L(v_6)$ is either 13 or 19. If the former, then $L(v_7)$ is either 19 or 20, leading to no available label for v_8. If the latter, then $L(v_7) = 13$, again leading to no available label for v_8. Hence $i_0 \neq 2r - 4$. □

5. Concluding remarks

We have shown in Section 4 that in the cases $r = 3$ and $r = 4$, the upper bounds given in Theorem 4.1 are sharp. We thus pose the following conjecture which would seem to require a proof that is lengthier and more case-driven than that of Theorem 4.7:

Conjecture 5.1. *For $r \geq 3$ and $1 \leq \frac{j}{k} \leq 2$,*

$$\lambda_{j,k}\big(P_\infty(r)\big) = \begin{cases} (2r-2)j & \text{if } 1 \leq \dfrac{j}{k} \leq \dfrac{r}{r-1} \\ 2rk & \text{if } \dfrac{r}{r-1} \leq \dfrac{j}{k} \leq 2. \end{cases}$$

We also note that in the case $\frac{j}{k} \geq 2$, the $\lambda_{j,k}$-number of $P_n(r)$ has been determined in Section 2 not only for infinite n but for all finite n as well. In the case $1 \leq \frac{j}{k} \leq 2$, we have derived (but not presented) $\lambda_{j,k}\big(P_n(r)\big)$ for $n = 2r$ and $n = 2r + 1$. But unfortunately, results for larger finite n have eluded us.

We mention that in the spirit of Griggs and Jin [6], the $\lambda_{j,k}$-number of $P_n(r)$ for $0 < \frac{j}{k} < 1$ bears exploration.

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